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A FAMILY OF DISCONTINUOUS-GALERKIN-BASED VARIATIONAL TIME INTEGRATORS

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Variational integrators (VI) provide a way to design structure-preserving time integrators for problems whose dynamics are generated by a Hamiltonian. The idea is to obtain the algorithm from discretizing Hamilton's variational principle [1]. In this way, the computed discrete trajectories are stationary points of a discrete action functional. As a result, the discrete trajectories approximate the exact trajectories of the system, and preserve the energy of the system for long times.

This abstract concentrates on extending the ideas of constructing (continuous) Galerkin variational integrator in order to build higher order discontinuous Galerkin variational integrators (DG-VI) for finite-dimensional systems. The resulting DG time integrators should correspond to a family of symplectic-discontinuous Runge-Kutta methods. They fall in the category of structure-preserving methods because (i) they (should be) are symplectic, (ii) they (should) conserve the invariants associated to symmetries of the Lagrangian and (iii) they (should) present an excellent long term energy behavior (due to the existence of a shadow Hamiltonian).

More precisely, we consider a finite-dimensional system described by generalized coordinates q belonging to an appropriate configuration manifold Q. We assume that the dynamics of such a system is characterized by the Lagrangian function $\mathsf{L}(q, \dot{q})$ which in many cases may be obtained as the sum of the kinetic plus the potential energies.

Let $q : [0, T] \to \mathbb{R}^m$, $m \in \mathbb{N}$, be the trajectory we look for. The Hamiltonian dynamics amounts to find the stationary point of the action integral,

$$S[q] := \int_0^T L(q, \dot{q})\, dt,$$

where

$$L(q, \dot{q}) := \frac{1}{2}\dot{q}M\dot{q} - V(q)$$

The Euler-Lagrange equation is given by

$$M\ddot{q} = -\nabla V(q), \tag{1}$$

The main part of this work is developing numerical approximations to (1) by defining an extension of the time derivative, the DG derivative (D_{DG}) [2],

$$D_{DG}q := \frac{d_h q}{dt} + r\left(\llbracket q \rrbracket\right),$$

where $\frac{d_h}{dt}$ means differentiation within each time interval where q is continuous, and r is a lifting operator on the jump of q at each time step t_k ($\llbracket q \rrbracket = q_k^- - q_k^+$). q has well-defined values $q_k^\pm$ at all $t_k^\pm$ (nearby t_k).

Then, the general expression of the discrete action integral S_d is given by

$$S_d(q) = \int_0^T \, dt = \frac{1}{2}\int_0^T M(D_{DG}q)^2\, dt - \int_0^T V(q)\, dt.$$

Considering a linear interpolation for q, the algorithm in the form of Hamiltonian dynamics reads,

$$\begin{aligned}
p^{k-} &= -D_1 L_d\left(q^{k-}, q^{k+}, q^{(k+1)-}, q^{(k+1)+}, \Delta t\right),\\
p^{k+} &= -D_2 L_d\left(q^{k-}, q^{k+}, q^{(k+1)-}, q^{(k+1)+}, \Delta t\right),\\
p^{(k+1)-} &= D_3 L_d\left(q^{k-}, q^{k+}, q^{(k+1)-}, q^{(k+1)+}, \Delta t\right),\\
p^{(k+1)+} &= D_4 L_d\left(q^{k-}, q^{k+}, q^{(k+1)-}, q^{(k+1)+}, \Delta t\right),
\end{aligned}$$

where L_d is the approximation of action integral, and $p^{k\pm}$ are the values of momentum at all $t_k^\pm$. To initiate the process, we provide the values for

$$q^{0-}, \quad q^{0+}, \quad p^{0-}, \quad \text{and} \quad p^{0+}.$$

The method can also be applied for higher orders with no restriction. The numerical performance of the algorithm is examined through some examples.

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